Übungsblatt 8

Integralrechnung im $\mathbb{R}^n$

1. Doppelintegrale

1.1. Doppelintegrale mit konstanten Integrationsgrenzen

Berechnen Sie die folgenden Doppelintegrale

A1

Beispiel 1:

$$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy$$

Innere Integration nach x: $$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy = 3 \int_{0}^{1} (1 + y^2)\, dy = 4$$

Innere Integration nach y: $$I = \int_{y=0}^{1} \int_{x=-2}^{1} (x^2 + y^2)\, dx\, dy = \int_{-2}^{1} \left(x^2 + \frac{1}{3}\right) dx = 4$$

Aufgaben:

a) $I_1 = \int_{x=0}^{1} \int_{y=0}^{1} (x^2 + y)\, dx\, dy,\quad I_2 = \int_{x=0}^{1} \int_{y=0}^{3} \left(\sqrt{x} + \sqrt{y+1}\right) dx\, dy,\quad I_3 = \int_{x=0}^{1} \int_{y=0}^{1} \sqrt{xy}\, dx\, dy$

b) $I_1 = \int_{x=0}^{2} \int_{y=0}^{\pi} x \sin y\, dx\, dy,\quad I_2 = \int_{x=0}^{3} \int_{y=0}^{\pi} x^2 \sin y\, dx\, dy,\quad I_3 = \int_{x=0}^{1} \int_{y=0}^{\pi/4} x \cos(2y)\, dx\, dy$

c) $I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(x + y)\, dx\, dy,\quad I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \cos(x + y)\, dx\, dy,\quad I_3 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} x \cos(x + y)\, dx\, dy$

A2

$$a)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\, \cos(2y)\, dx\, dy, \qquad I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(2x)\, \cos(3y)\, dx\, dy$$

$$b)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\, \cos^2 y\, dx\, dy, \qquad I_2 = \int_{x=0}^{\pi/4}\int_{y=0}^{\pi/2} \sin(2x)\, \cos^2 y\, dx\, dy$$

$$c)\quad I_1 = \int_{x=0}^{\pi}\int_{y=1}^{2} y \cdot \cos(xy)\, dx\, dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\, \sin(xy)\, dx\, dy$$

$$d)\quad I_1 = \int_{x=1}^{3}\int_{y=1}^{2} x\, \ln(xy)\, dx\, dy, \qquad I_2 = \int_{x=1}^{3}\int_{y=1}^{2} x^2\, \ln(xy)\, dx\, dy$$

$$e)\quad I_1 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\sin y}{x}\, dx\, dy, \qquad I_2 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\cos y}{x^2}\, dx\, dy, \qquad I_3 = \int_{x=1}^{3}\int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3}\, dx\, dy$$

A3

$$a)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} e^{x-2y}\, dx\, dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{1} y^2\, e^{x+2}\, dx\, dy$$

$$b)\quad I_1 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\, e^x}{y}\, dx\, dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=1}^{3} \frac{x\, e^x}{y^2}\, dx\, dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\, e^{2x}}{y^3}\, dx\, dy$$

$$c)\quad I_1 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{2x}{y} - \frac{y}{x}\right) dx\, dy, \qquad I_2 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{x}{y} - \frac{y^2}{x^2}\right) dx\, dy$$

$$d)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} \frac{x}{1+xy}\, dx\, dy, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x}{1+2xy}\, dx\, dy, \qquad I_3 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x^2}{1+xy}\, dx\, dy$$

$$e)\quad I_1 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{1+y}\, dx\, dy, \qquad I_2 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{(1+y)^2}\, dx\, dy$$

1.2. Doppelintegrale mit beliebigen Integrationsgrenzen

A4

$$a)\quad I_1 = \int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy, \qquad I_2 = \int_{y=0}^{2}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=0}^{\sqrt{4-x^2}} xy\,dy\,dx$$

$$b)\quad I_1 = \int_{x=0}^{3}\int_{y=0}^{x} x\,y^2\,dy\,dx, \qquad I_2 = \int_{x=0}^{2}\int_{y=0}^{x} x^2\,y^2\,dy\,dx, \qquad I_3 = \int_{x=0}^{1}\int_{y=1-x}^{1-x^2} xy\,dy\,dx$$

$$c)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{x} (x^2+y^2)\,dy\,dx, \qquad I_2 = \int_{y=0}^{3}\int_{x=0}^{\sqrt{y}} (x^3+y^3)\,dx\,dy, \qquad I_3 = \int_{x=0}^{1}\int_{y=0}^{x^2-1} (x+y)\,dy\,dx$$

$$d)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (1+\sin y)\,dy\,dx, \qquad I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (\cos x + \sin y)\,dy\,dx$$

1.3. Doppelintegrale in Polarkoordinaten

Berechnen Sie die folgenden Doppelintegrale und zeichnen Sie den Integrationsbereich

A5

$$I_1 = \iint_A xy\,dx\,dy, \qquad A:\quad 1 \leqslant r \leqslant 3, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{4}$$

$$I_2 = \iint_A y^2\,\sqrt{4-x^2-y^2}\,dx\,dy, \qquad A:\quad x^2+y^2 \leqslant 4, \quad y \geqslant 0$$

Berechnen Sie die folgenden Doppelintegrale

A6

$$a)\quad I_1 = \iint_A x^2\,e^{-(x^2+y^2)}\,dx\,dy, \qquad A = x^2+y^2 \leqslant 1$$

$$b)\quad I_1 = \int_{\varphi=0}^{\pi/2}\int_{r=0}^{\cos^2\varphi} r\,dr\,d\varphi,$$

1.4. Doppelintegrale in der Volumenberechnung

Berechnen Sie die Volumina der Körper, die durch folgende Flächen begrenzt werden oder durch andere Angaben bestimmt werden

A7

$a)$ $f(x, y) = 2 + \sin x \cdot \sin y, \qquad A_f: \quad -\pi \leqslant x, y \leqslant \pi$

$\quad g(x, y) = 2 + \sin x \cdot \sin y, \qquad A_g: \quad x^2 + y^2 \leqslant \pi^2$

$b)$ $x^2 + y^2 = 9, \quad z = 0, \quad z = 9 - y$

$c)$ $y = x^2, \quad y = 4, \quad z = 3 + x + 2y$

$d)$ $z = 2 - 2x - y, \quad x = 0, \quad y = 0, \quad z = 0$

Lösungen – Übungsblatt 8 :

1.

Doppelintegrale mit konstanten Integrationsgrenzen

$$a)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{1} (x^2+y)\,dx\,dy = \int_0^1 \left(\frac{1}{3}+y\right) dy = \frac{5}{6}$$

$$I_2 = \int_{x=0}^{1}\int_{y=0}^{3} \left(\sqrt{x}+\sqrt{y+1}\right) dx\,dy = \int_0^3 \left(\frac{2}{3}+\sqrt{y+1}\right) dy = \frac{20}{3}$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{1} \sqrt{xy}\,dx\,dy = \frac{2}{3}\int_0^1 \sqrt{y}\,dy = \frac{4}{9}$$

$$b)\quad I_1 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\,\sin y\,dx\,dy = 2\int_0^{\pi} \sin y\,dy = 4$$

$$I_2 = \int_{x=0}^{3}\int_{y=0}^{\pi} x^2\,\sin y\,dx\,dy = 9\int_0^{\pi} \sin y\,dy = 18$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{\pi/4} x\,\cos(2y)\,dx\,dy = \frac{1}{2}\int_0^{\pi/4} \cos(2y)\,dy = \frac{1}{4}$$

$$c)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(x+y)\,dx\,dy = \int_0^{\pi/2} (\sin y+\cos y)\,dy = 2$$

$$I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \cos(x+y)\,dx\,dy = \int_0^{\pi/2} (-\sin y+\cos y)\,dy = 0$$

$$I_3 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} x\,\cos(x+y)\,dx\,dy = \int_0^{\pi/2} (-x\,\sin x + x\,\cos x)\,dx = -2+\frac{\pi}{2}$$

Lsg.

2. ,

$$a)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\,\cos(2y)\,dx\,dy = \int_0^{\pi/2} \cos(2y)\,dy = 0$$

$$I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin(2x)\,\cos(3y)\,dx\,dy = \int_0^{\pi/2} \cos(3y)\,dy = -\frac{1}{3}$$

$$b)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2} \sin x\,\cos^2 y\,dx\,dy = \int_0^{\pi/2} \cos^2 y\,dy = \frac{\pi}{4}$$

$$I_2 = \int_{x=0}^{\pi/4}\int_{y=0}^{\pi/2} \sin(2x)\,\cos^2 y\,dx\,dy = \frac{1}{2}\int_0^{\pi/2} \cos^2 y\,dy = \frac{\pi}{8}$$

$$c)\quad I_1 = \int_{x=0}^{\pi}\int_{y=1}^{2} y\cdot\cos(xy)\,dx\,dy = \int_1^{2} \sin(\pi y)\,dy = -\frac{2}{\pi}$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{\pi} x\,\sin(xy)\,dx\,dy = \int_0^{2} (1-\cos(\pi x))\,dx = 2$$

$$d)\quad I_1 = \int_{x=1}^{3}\int_{y=1}^{2} x\,\ln(xy)\,dx\,dy = \int_1^{3} (x\,\ln x - x + 2x\,\ln 2)\,dx = -6 + 8\ln 2 + \frac{9}{2}\ln 3 \simeq 4.49$$

$$I_2 = \int_{x=1}^{3}\int_{y=1}^{2} x^2\,\ln(xy)\,dx\,dy = \int_1^{3} \left(x^2\ln x - x^2 + 2x^2\ln 2\right)dx =$$

$$= -\frac{104}{9} + \frac{52}{3}\ln 2 + 9\ln 3 \simeq 10.37$$

$$e)\quad I_1 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\sin y}{x}\,dx\,dy = \int_1^{2} \frac{dx}{x} = \ln 2 \simeq 0.69$$

$$I_2 = \int_{x=1}^{2}\int_{y=0}^{\pi/2} \frac{\cos y}{x^2}\,dx\,dy = \int_1^{2} \frac{dx}{x^2} = \frac{1}{2}$$

$$I_3 = \int_{x=1}^{3}\int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3}\,dx\,dy = \frac{1}{2}\int_1^{3} \frac{dx}{x^3} = \frac{2}{9}$$

Lsg.

3.

a) $$I_1 = \int_{x=0}^{1}\int_{y=0}^{1} e^{x-2y}\,dx\,dy = \int_0^1 \left(e^{1-2y} - e^{-2y}\right) dy = \frac{1}{2}\left(-1 + e + e^{-2} - e^{-1}\right) \simeq 0.74$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{1} y^2\, e^{x+2}\,dy\,dx = \frac{1}{3}\int_0^2 e^{x+2}\,dx = \frac{1}{3}(e^4 - e^2) \simeq 15.75$$

b) $$I_1 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\,e^x}{y}\,dy\,dx = \ln 2 \int_0^1 x\,e^x\,dx = \ln 2 \simeq 0.69$$

$$I_2 = \int_{x=0}^{2}\int_{y=1}^{3} \frac{x\,e^x}{y^2}\,dy\,dx = \frac{2}{3}\int_0^2 x\,e^x\,dx = \frac{2}{3}\left(1 + e^2\right)$$

$$I_3 = \int_{x=0}^{1}\int_{y=1}^{2} \frac{x\,e^{2x}}{y^3}\,dy\,dx = \frac{3}{8}\int_0^1 x\,e^{2x}\,dx = \frac{3}{32}\left(1 + e^2\right)$$

c) $$I_1 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{2x}{y} - \frac{y}{x}\right) dy\,dx = \frac{1}{2}\int_1^2 \left(4\,x\ln 2 - \frac{3}{x}\right) dx = \frac{3}{2}\ln 2 \simeq 1.04$$

$$I_2 = \int_{x=1}^{2}\int_{y=1}^{2} \left(\frac{x}{y} - \frac{y^2}{x^2}\right) dy\,dx = \frac{1}{3}\int_1^2 \left(3\,\ln 2\cdot x - \frac{7}{x^2}\right) dx = -\frac{7}{6} + \frac{3}{2}\ln 2 \simeq -0.13$$

d) $$I_1 = \int_{x=0}^{1}\int_{y=0}^{1} \frac{x}{1+xy}\,dy\,dx = \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1 \simeq 0.39$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x}{1+2xy}\,dy\,dx = \frac{1}{2}\int_0^2 \ln(1+2x)\,dx = \frac{5}{4}\ln 5 - 1 \simeq 1.01$$

$$I_3 = \int_{x=0}^{2}\int_{y=0}^{1} \frac{x^2}{1+xy}\,dy\,dx = \int_0^2 x\,\ln(1+x)\,dx = \int_1^3 (u-1)\ln u\,du = \frac{3}{2}\ln 3 \simeq 1.65 \quad (u = 1+x)$$

e) $$I_1 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{1+y}\,dy\,dx = \ln 2\int_0^4 \sqrt{x}\,dx = \frac{16}{3}\ln 2 \simeq 3.70$$

$$I_2 = \int_{x=0}^{4}\int_{y=0}^{1} \frac{\sqrt{x}}{(1+y)^2}\,dy\,dx = \frac{1}{2}\int_0^4 \sqrt{x}\,dx = \frac{8}{3}$$

Lsg.

4. Doppelintegrale mit beliebigen Integrationsgrenzen

$$a)\quad I_1 = \int_{y=0}^{1}\int_{x=0}^{y} xy\,dx\,dy = \frac{1}{2}\int_0^1 y^3\,dy = \frac{1}{8}$$

$$I_2 = \int_{y=0}^{2}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy = \frac{1}{2}\int_0^2 y^2\,dy = \frac{4}{3}$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{\sqrt{4-x^2}} xy\,dx\,dy = \frac{1}{2}\int_0^1 x\,(4-x^2)\,dx = \frac{7}{8}$$

$$b)\quad I_1 = \int_{x=0}^{3}\int_{y=0}^{x} x\,y^2\,dx\,dy = \frac{1}{3}\int_0^3 x^4\,dx = \frac{81}{5}$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{x} x^2 y^2\,dx\,dy = \frac{1}{3}\int_0^2 x^5\,dx = \frac{32}{9}$$

$$I_3 = \int_{x=0}^{1}\int_{y=1-x}^{1-x^2} xy\,dx\,dy = \frac{1}{2}\int_0^1 x\,\left((1-x^2)^2-(1-x)^2\right)\,dx = \frac{1}{24}$$

$$c)\quad I_1 = \int_{x=0}^{1}\int_{y=0}^{x} (x^2+y^2)\,dx\,dy = \frac{4}{3}\int_0^1 x^3\,dx = \frac{1}{3}$$

$$I_2 = \int_{y=0}^{3}\int_{x=0}^{\sqrt{y}} (x^3+y^3)\,dx\,dy = \int_0^3 \left(\frac{y^2}{4}+y^{7/2}\right)dy = \frac{9}{4}+18\sqrt{3} \simeq 33.43$$

$$I_3 = \int_{x=0}^{1}\int_{y=0}^{x^2-1} (x+y)\,dx\,dy = \int_0^1 \left(x\,(x^2-1)+\frac{1}{2}(x^2-1)^2\right)dx = \frac{1}{60}$$

$$d)\quad I_1 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (1+\sin y)\,dx\,dy = \int_0^{\pi/2} (1+x-\cos x)\,dx = \frac{\pi^2}{8}+\frac{\pi}{2}-1 = 1.8$$

$$I_2 = \int_{x=0}^{\pi/2}\int_{y=0}^{x} (\cos x+\sin y)\,dx\,dy = \int_0^{\pi/2} (1+x\cos x-\cos x)\,dx = -2+\pi \simeq 1.14$$

Lsg.

5. Doppelintegrale in Polarkoordinaten

$$I_1 = \iint_A xy\,dx\,dy = \frac{1}{2}\int_{r=1}^{3}\int_{\varphi=0}^{\pi/4} r^3 \sin(2\varphi)dr\,d\varphi = 10\int_{\varphi=0}^{\pi/4}\sin(2\varphi)d\varphi = 5$$

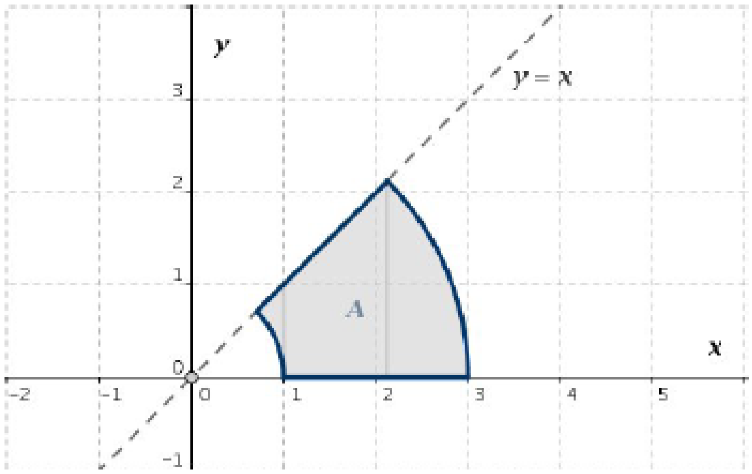

Abbildung 1: Darstellung des Integrationsbereiches für das Integral I_1 $A:\ 1 \leqslant r \leqslant 3,\quad 0 \leqslant \varphi \leqslant \frac{\pi}{4}$

$$I_2 = \int_A y^2\sqrt{4-x^2-y^2}\,dx\,dy = \int_{r=0}^{2} r^3\sqrt{4-r^2}\,dr\int_{\varphi=0}^{\pi}\sin^2\varphi\,d\varphi = \frac{\pi}{2}\int_{r=0}^{2} r^3\sqrt{4-r^2}\,dr = \frac{32}{15}\pi\, simeq\, 6.702$$

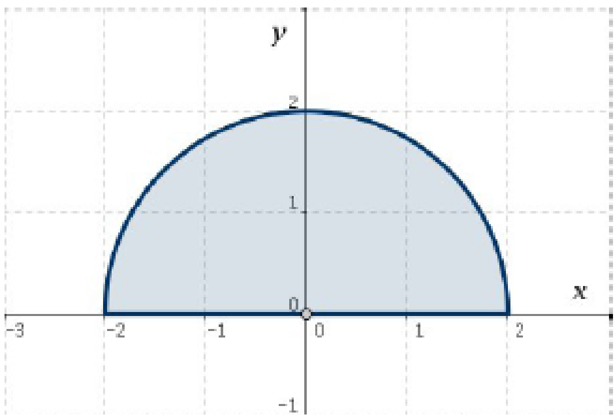

Abbildung 2: Darstellung des Integrationsbereiches für das Integral I_2 $A:\ 0 \leqslant r \leqslant 2,\quad 0 \leqslant \varphi \leqslant \pi$

6.

$$a)\quad I_1 = \iint_A x^2 e^{-(x^2+y^2)}\,dx\,dy = \int_{\varphi=0}^{2\pi}\cos^2\varphi\,d\varphi\int_{r=0}^{1} r^3 e^{-r^2}\,dr = \pi\int_{r=0}^{1} r^3 e^{-r^2}\,dr = \pi\left(\frac{1}{2}-e^{-1}\right) \simeq 0.415$$

$$b)\quad I_1 = \int_{\varphi=0}^{\pi/2}\int_{r=0}^{\cos^2\varphi} r\,dr\,d\varphi = \frac{1}{2}\int_{\varphi=0}^{\pi/2}\cos^4\varphi d\varphi = \frac{1}{16}\int_{\varphi=0}^{\pi/2}(\cos(4\varphi)+4\cos(2\varphi)+3)\,d\varphi = \frac{3}{32}\pi \simeq 0.295$$

7. Doppelintegrale in der Volumenberechnung

$a)$ $f(x,y) = 2 + \sin x \cdot \sin y, \qquad A_f: \; -\pi \leqslant x, y \leqslant \pi$

$$V = \iint\limits_A (2 + \sin x \cdot \sin y)\, dx\, dy = \int\limits_{x=-\pi}^{\pi} \int\limits_{y=-\pi}^{\pi} (2 + \sin x \cdot \sin y)\, dx\, dy = 4\pi \int\limits_{y=-\pi}^{\pi} dy = 8\pi^2 \simeq 78.96 \text{ VE}$$

$g(x,y) = 2 + \sin x \cdot \sin y, \qquad A_g: \; x^2 + y^2 \leqslant \pi^2$

$$V = \iint\limits_A (2 + \sin x \cdot \sin y)\, dx\, dy = \int\limits_{x=-\pi}^{\pi} \int\limits_{y=-\sqrt{\pi^2-x^2}}^{\sqrt{\pi^2-x^2}} (2 + \sin x \cdot \sin y)\, dy\, dx =$$

$$= 4 \int\limits_{x=-\pi}^{\pi} \sqrt{\pi^2 - x^2}\, dx = 2\pi^3 \simeq 62.01 \text{ VE}$$

$b)$ $x^2 + y^2 = 9, \qquad z = 0, \qquad z = 9 - y$

$$V = \iint\limits_A (9 - y)\, dx\, dy = 81\pi \simeq 254.47 \text{ VE}$$

$c)$ $y = x^2, \quad y = 4, \quad z = 3 + x + 2y$

$$V = \iint\limits_A (3 + x + 2y)\, dx dy = \int\limits_{x=-2}^{2} \int\limits_{y=x^2}^{4} (3 + x + 2y)\, dy\, dx = \frac{416}{5} \simeq 83.2 \text{ VE}$$

$d)$ $z = 2 - 2x - y, \qquad x = 0, \qquad y = 0, \qquad z = 0, \qquad V = \dfrac{2}{3}$