

Übungsblatt 8

Integralrechnung im $\mathbb{R}^n$

1. Doppelintegrale

1.1. Doppelintegrale mit konstanten Integrationsgrenzen

Berechnen Sie die folgenden Doppelintegrale

A1

Beispiel 1:

$$I = \int_{y=0}^1 \int_{x=-2}^1 (x^2 + y^2) dx dy$$

Innere Integration nach x :
$$I = \int_{y=0}^1 \int_{x=-2}^1 (x^2 + y^2) dx dy = 3 \int_0^1 (1 + y^2) dy = 4$$

Innere Integration nach y :
$$I = \int_{y=0}^1 \int_{x=-2}^1 (x^2 + y^2) dx dy = \int_{-2}^1 \left(x^2 + \frac{1}{3} \right) dx = 4$$

Aufgaben:

a) $I_1 = \int_{x=0}^1 \int_{y=0}^1 (x^2 + y) dx dy,$ $I_2 = \int_{x=0}^1 \int_{y=0}^3 (\sqrt{x} + \sqrt{y+1}) dx dy,$ $I_3 = \int_{x=0}^1 \int_{y=0}^1 \sqrt{xy} dx dy$

b) $I_1 = \int_{x=0}^2 \int_{y=0}^{\pi} x \sin y dx dy,$ $I_2 = \int_{x=0}^3 \int_{y=0}^{\pi} x^2 \sin y dx dy,$ $I_3 = \int_{x=0}^1 \int_{y=0}^{\pi/4} x \cos(2y) dx dy$

c) $I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(x+y) dx dy,$ $I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \cos(x+y) dx dy,$ $I_3 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} x \cos(x+y) dx dy$

A2

$$\begin{aligned}
 a) \quad I_1 &= \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin x \cos(2y) dx dy, & I_2 &= \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(2x) \cos(3y) dx dy \\
 b) \quad I_1 &= \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin x \cos^2 y dx dy, & I_2 &= \int_{x=0}^{\pi/4} \int_{y=0}^{\pi/2} \sin(2x) \cos^2 y dx dy \\
 c) \quad I_1 &= \int_{x=0}^{\pi} \int_{y=1}^2 y \cdot \cos(xy) dx dy, & I_2 &= \int_{x=0}^2 \int_{y=0}^{\pi} x \sin(xy) dx dy \\
 d) \quad I_1 &= \int_{x=1}^3 \int_{y=1}^2 x \ln(xy) dx dy, & I_2 &= \int_{x=1}^3 \int_{y=1}^2 x^2 \ln(xy) dx dy \\
 e) \quad I_1 &= \int_{x=1}^2 \int_{y=0}^{\pi/2} \frac{\sin y}{x} dx dy, & I_2 &= \int_{x=1}^2 \int_{y=0}^{\pi/2} \frac{\cos y}{x^2} dx dy, & I_3 &= \int_{x=1}^3 \int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3} dx dy
 \end{aligned}$$

A3

$$\begin{aligned}
 a) \quad I_1 &= \int_{x=0}^1 \int_{y=0}^1 e^{x-2y} dx dy, & I_2 &= \int_{x=0}^2 \int_{y=0}^1 y^2 e^{x+2} dx dy \\
 b) \quad I_1 &= \int_{x=0}^1 \int_{y=1}^2 \frac{x e^x}{y} dx dy, & I_2 &= \int_{x=0}^2 \int_{y=1}^3 \frac{x e^x}{y^2} dx dy, & I_3 &= \int_{x=0}^1 \int_{y=1}^2 \frac{x e^{2x}}{y^3} dx dy \\
 c) \quad I_1 &= \int_{x=1}^2 \int_{y=1}^2 \left(\frac{2x}{y} - \frac{y}{x} \right) dx dy, & I_2 &= \int_{x=1}^2 \int_{y=1}^2 \left(\frac{x}{y} - \frac{y^2}{x^2} \right) dx dy \\
 d) \quad I_1 &= \int_{x=0}^1 \int_{y=0}^1 \frac{x}{1+xy} dx dy, & I_2 &= \int_{x=0}^2 \int_{y=0}^1 \frac{x}{1+2xy} dx dy, & I_3 &= \int_{x=0}^2 \int_{y=0}^1 \frac{x^2}{1+xy} dx dy \\
 e) \quad I_1 &= \int_{x=0}^4 \int_{y=0}^1 \frac{\sqrt{x}}{1+y} dx dy, & I_2 &= \int_{x=0}^4 \int_{y=0}^1 \frac{\sqrt{x}}{(1+y)^2} dx dy
 \end{aligned}$$

1.2. Doppelintegrale mit beliebigen Integrationsgrenzen

A4

$$a) \quad I_1 = \int_{y=0}^1 \int_{x=0}^y xy \, dx \, dy, \quad I_2 = \int_{y=0}^2 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy, \quad I_3 = \int_{x=0}^1 \int_{y=0}^{\sqrt{4-x^2}} xy \, dy \, dx$$

$$b) \quad I_1 = \int_{x=0}^3 \int_{y=0}^x xy^2 \, dy \, dx, \quad I_2 = \int_{x=0}^2 \int_{y=0}^x x^2 y^2 \, dy \, dx, \quad I_3 = \int_{x=0}^1 \int_{y=1-x}^{1-x^2} xy \, dy \, dx$$

$$c) \quad I_1 = \int_{x=0}^1 \int_{y=0}^x (x^2 + y^2) \, dy \, dx, \quad I_2 = \int_{y=0}^3 \int_{x=0}^{\sqrt{y}} (x^3 + y^3) \, dx \, dy, \quad I_3 = \int_{x=0}^1 \int_{y=0}^{x^2-1} (x+y) \, dy \, dx$$

$$d) \quad I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^x (1 + \sin y) \, dy \, dx, \quad I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^x (\cos x + \sin y) \, dy \, dx$$

1.3. Doppelintegrale in Polarkoordinaten

Berechnen Sie die folgenden Doppelintegrale und zeichnen Sie den Integrationsbereich

A5

$$I_1 = \iint_A xy \, dx \, dy, \quad A: 1 \leq r \leq 3, \quad 0 \leq \varphi \leq \frac{\pi}{4}$$

$$I_2 = \iint_A y^2 \sqrt{4-x^2-y^2} \, dx \, dy, \quad A: x^2 + y^2 \leq 4, \quad y \geq 0$$

Berechnen Sie die folgenden Doppelintegrale

A6

$$a) \quad I_1 = \iint_A x^2 e^{-(x^2+y^2)} \, dx \, dy, \quad A = x^2 + y^2 \leq 1$$

$$b) \quad I_1 = \int_{\varphi=0}^{\pi/2} \int_{r=0}^{\cos^2 \varphi} r \, dr \, d\varphi,$$

1.4. Doppelintegrale in der Volumenberechnung

Berechnen Sie die Volumina der Körper, die durch folgende Flächen begrenzt werden oder durch andere Angaben bestimmt werden

A7

$$a) \quad f(x, y) = 2 + \sin x \cdot \sin y, \quad A_f : -\pi \leq x, y \leq \pi$$

$$g(x, y) = 2 + \sin x \cdot \sin y, \quad A_g : x^2 + y^2 \leq \pi^2$$

$$b) \quad x^2 + y^2 = 9, \quad z = 0, \quad z = 9 - y$$

$$c) \quad y = x^2, \quad y = 4, \quad z = 3 + x + 2y$$

$$d) \quad z = 2 - 2x - y, \quad x = 0, \quad y = 0, \quad z = 0$$

Lösungen - Übungsblatt 8:

1.

Doppelintegrale mit konstanten Integrationsgrenzen

$$a) \quad I_1 = \int_{x=0}^1 \int_{y=0}^1 (x^2 + y) dx dy = \int_0^1 \left(\frac{1}{3} + y \right) dy = \frac{5}{6}$$

$$I_2 = \int_{x=0}^1 \int_{y=0}^3 (\sqrt{x} + \sqrt{y+1}) dx dy = \int_0^3 \left(\frac{2}{3} + \sqrt{y+1} \right) dy = \frac{20}{3}$$

$$I_3 = \int_{x=0}^1 \int_{y=0}^1 \sqrt{xy} dx dy = \frac{2}{3} \int_0^1 \sqrt{y} dy = \frac{4}{9}$$

$$b) \quad I_1 = \int_{x=0}^2 \int_{y=0}^{\pi} x \sin y dx dy = 2 \int_0^{\pi} \sin y dy = 4$$

$$I_2 = \int_{x=0}^3 \int_{y=0}^{\pi} x^2 \sin y dx dy = 9 \int_0^{\pi} \sin y dy = 18$$

$$I_3 = \int_{x=0}^1 \int_{y=0}^{\pi/4} x \cos(2y) dx dy = \frac{1}{2} \int_0^{\pi/4} \cos(2y) dy = \frac{1}{4}$$

$$c) \quad I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(x+y) dx dy = \int_0^{\pi/2} (\sin y + \cos y) dy = 2$$

$$I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \cos(x+y) dx dy = \int_0^{\pi/2} (-\sin y + \cos y) dy = 0$$

$$I_3 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} x \cos(x+y) dx dy = \int_0^{\pi/2} (-x \sin x + x \cos x) dx = -2 + \frac{\pi}{2}$$

Lsg.
2. ,

$$a) \quad I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin x \cos(2y) dx dy = \int_0^{\pi/2} \cos(2y) dy = 0$$

$$I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin(2x) \cos(3y) dx dy = \int_0^{\pi/2} \cos(3y) dy = -\frac{1}{3}$$

$$b) \quad I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} \sin x \cos^2 y dx dy = \int_0^{\pi/2} \cos^2 y dy = \frac{\pi}{4}$$

$$I_2 = \int_{x=0}^{\pi/4} \int_{y=0}^{\pi/2} \sin(2x) \cos^2 y dx dy = \frac{1}{2} \int_0^{\pi/2} \cos^2 y dy = \frac{\pi}{8}$$

$$c) \quad I_1 = \int_{x=0}^{\pi} \int_{y=1}^2 y \cdot \cos(xy) dx dy = \int_1^2 \sin(\pi y) dy = -\frac{2}{\pi}$$

$$I_2 = \int_{x=0}^2 \int_{y=0}^{\pi} x \sin(xy) dx dy = \int_0^2 (1 - \cos(\pi x)) dx = 2$$

$$d) \quad I_1 = \int_{x=1}^3 \int_{y=1}^2 x \ln(xy) dx dy = \int_1^3 (x \ln x - x + 2x \ln 2) dx = -6 + 8 \ln 2 + \frac{9}{2} \ln 3 \simeq 4.49$$

$$I_2 = \int_{x=1}^3 \int_{y=1}^2 x^2 \ln(xy) dx dy = \int_1^3 (x^2 \ln x - x^2 + 2x^2 \ln 2) dx =$$

$$= -\frac{104}{9} + \frac{52}{3} \ln 2 + 9 \ln 3 \simeq 10.37$$

$$e) \quad I_1 = \int_{x=1}^2 \int_{y=0}^{\pi/2} \frac{\sin y}{x} dx dy = \int_1^2 \frac{dx}{x} = \ln 2 \simeq 0.69$$

$$I_2 = \int_{x=1}^2 \int_{y=0}^{\pi/2} \frac{\cos y}{x^2} dx dy = \int_1^2 \frac{dx}{x^2} = \frac{1}{2}$$

$$I_3 = \int_{x=1}^3 \int_{y=0}^{\pi/4} \frac{\cos(2y)}{x^3} dx dy = \frac{1}{2} \int_1^3 \frac{dx}{x^3} = \frac{2}{9}$$

Lsg.

3.

$$a) \quad I_1 = \int_{x=0}^1 \int_{y=0}^1 e^{x-2y} dx dy = \int_0^1 (e^{1-2y} - e^{-2y}) dy = \frac{1}{2} (-1 + e + e^{-2} - e^{-1}) \simeq 0.74$$

$$I_2 = \int_{x=0}^2 \int_{y=0}^1 y^2 e^{x+2} dy dx = \frac{1}{3} \int_0^2 e^{x+2} dx = \frac{1}{3} (e^4 - e^2) \simeq 15.75$$

$$b) \quad I_1 = \int_{x=0}^1 \int_{y=1}^2 \frac{x e^x}{y} dy dx = \ln 2 \int_0^1 x e^x dx = \ln 2 \simeq 0.69$$

$$I_2 = \int_{x=0}^2 \int_{y=1}^3 \frac{x e^x}{y^2} dy dx = \frac{2}{3} \int_0^2 x e^x dx = \frac{2}{3} (1 + e^2)$$

$$I_3 = \int_{x=0}^1 \int_{y=1}^2 \frac{x e^{2x}}{y^3} dy dx = \frac{3}{8} \int_0^1 x e^{2x} dx = \frac{3}{32} (1 + e^2)$$

$$c) \quad I_1 = \int_{x=1}^2 \int_{y=1}^2 \left(\frac{2x}{y} - \frac{y}{x} \right) dy dx = \frac{1}{2} \int_1^2 \left(4x \ln 2 - \frac{3}{x} \right) dx = \frac{3}{2} \ln 2 \simeq 1.04$$

$$I_2 = \int_{x=1}^2 \int_{y=1}^2 \left(\frac{x}{y} - \frac{y^2}{x^2} \right) dy dx = \frac{1}{3} \int_1^2 \left(3 \ln 2 \cdot x - \frac{7}{x^2} \right) dx = -\frac{7}{6} + \frac{3}{2} \ln 2 \simeq -0.13$$

$$d) \quad I_1 = \int_{x=0}^1 \int_{y=0}^1 \frac{x}{1+xy} dy dx = \int_0^1 \ln(1+x) dx = 2 \ln 2 - 1 \simeq 0.39$$

$$I_2 = \int_{x=0}^2 \int_{y=0}^1 \frac{x}{1+2xy} dy dx = \frac{1}{2} \int_0^2 \ln(1+2x) dx = \frac{5}{4} \ln 5 - 1 \simeq 1.01$$

$$I_3 = \int_{x=0}^2 \int_{y=0}^1 \frac{x^2}{1+xy} dy dx = \int_0^2 x \ln(1+x) dx = \int_1^3 (u-1) \ln u du = \frac{3}{2} \ln 3 \simeq 1.65 \quad (u = 1+x)$$

$$e) \quad I_1 = \int_{x=0}^4 \int_{y=0}^1 \frac{\sqrt{x}}{1+y} dy dx = \ln 2 \int_0^4 \sqrt{x} dx = \frac{16}{3} \ln 2 \simeq 3.70$$

$$I_2 = \int_{x=0}^4 \int_{y=0}^1 \frac{\sqrt{x}}{(1+y)^2} dy dx = \frac{1}{2} \int_0^4 \sqrt{x} dx = \frac{8}{3}$$

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4. Doppelintegrale mit beliebigen Integrationsgrenzen

$$a) \quad I_1 = \int_{y=0}^1 \int_{x=0}^y xy \, dx \, dy = \frac{1}{2} \int_0^1 y^3 \, dy = \frac{1}{8}$$

$$I_2 = \int_{y=0}^2 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy = \frac{1}{2} \int_0^2 y^2 \, dy = \frac{4}{3}$$

$$I_3 = \int_{x=0}^1 \int_{y=0}^{\sqrt{4-x^2}} xy \, dx \, dy = \frac{1}{2} \int_0^1 x(4-x^2) \, dx = \frac{7}{8}$$

$$b) \quad I_1 = \int_{x=0}^3 \int_{y=0}^x xy^2 \, dx \, dy = \frac{1}{3} \int_0^3 x^4 \, dx = \frac{81}{5}$$

$$I_2 = \int_{x=0}^2 \int_{y=0}^x x^2 y^2 \, dx \, dy = \frac{1}{3} \int_0^2 x^5 \, dx = \frac{32}{9}$$

$$I_3 = \int_{x=0}^1 \int_{y=1-x}^{1-x^2} xy \, dx \, dy = \frac{1}{2} \int_0^1 x \left((1-x^2)^2 - (1-x)^2 \right) dx = \frac{1}{24}$$

$$c) \quad I_1 = \int_{x=0}^1 \int_{y=0}^x (x^2 + y^2) \, dx \, dy = \frac{4}{3} \int_0^1 x^3 \, dx = \frac{1}{3}$$

$$I_2 = \int_{y=0}^3 \int_{x=0}^{\sqrt{y}} (x^3 + y^3) \, dx \, dy = \int_0^3 \left(\frac{y^2}{4} + y^{7/2} \right) dy = \frac{9}{4} + 18\sqrt{3} \simeq 33.43$$

$$I_3 = \int_{x=0}^1 \int_{y=0}^{x^2-1} (x+y) \, dx \, dy = \int_0^1 \left(x(x^2-1) + \frac{1}{2}(x^2-1)^2 \right) dx = \frac{1}{60}$$

$$d) \quad I_1 = \int_{x=0}^{\pi/2} \int_{y=0}^x (1 + \sin y) \, dx \, dy = \int_0^{\pi/2} (1 + x - \cos x) \, dx = \frac{\pi^2}{8} + \frac{\pi}{2} - 1 = 1.8$$

$$I_2 = \int_{x=0}^{\pi/2} \int_{y=0}^x (\cos x + \sin y) \, dx \, dy = \int_0^{\pi/2} (1 + x \cos x - \cos x) \, dx = -2 + \pi \simeq 1.14$$

Lsg.

5. Doppelintegrale in Polarkoordinaten

$$I_1 = \iint_A xy \, dx \, dy = \frac{1}{2} \int_{\varphi=0}^{\pi/4} \int_{r=1}^3 r^3 \sin(2\varphi) \, dr \, d\varphi = 10 \int_{\varphi=0}^{\pi/4} \sin(2\varphi) \, d\varphi = 5$$

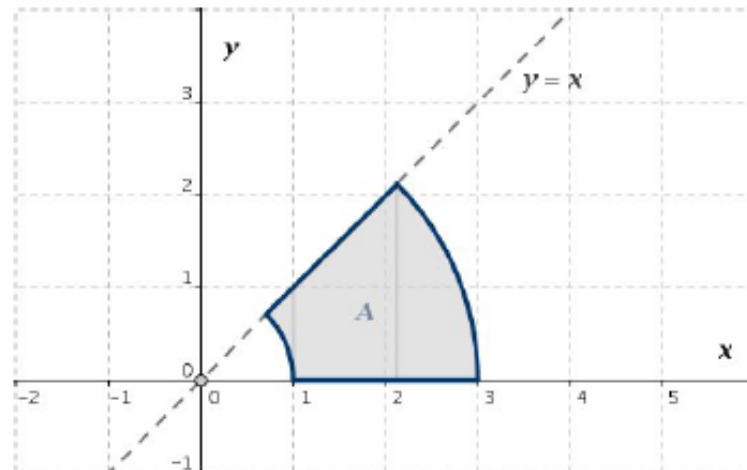


Abbildung 1: Darstellung des Integrationsbereiches für das Integral I_1 $A : 1 \leq r \leq 3, 0 \leq \varphi \leq \frac{\pi}{4}$

$$I_2 = \int_A y^2 \sqrt{4-x^2-y^2} \, dx \, dy = \int_{r=0}^2 r^3 \sqrt{4-r^2} \, dr \int_{\varphi=0}^{\pi} \sin^2 \varphi \, d\varphi = \frac{\pi}{2} \int_{r=0}^2 r^3 \sqrt{4-r^2} \, dr = \frac{32}{15} \pi \approx 6.702$$

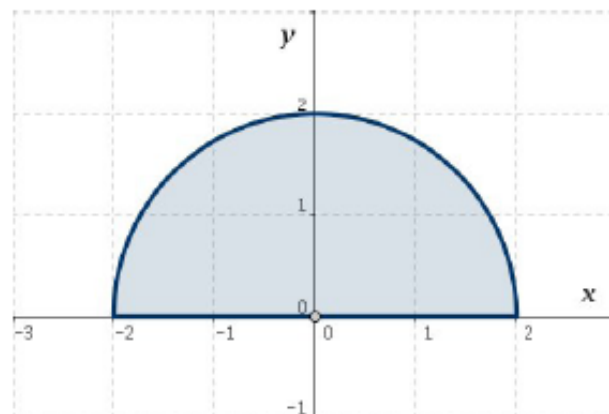


Abbildung 2: Darstellung des Integrationsbereiches für das Integral I_2 $A : 0 \leq r \leq 2, 0 \leq \varphi \leq \pi$

6.

$$a) \quad I_1 = \iint_A x^2 e^{-(x^2+y^2)} \, dx \, dy = \int_{\varphi=0}^{2\pi} \cos^2 \varphi \, d\varphi \int_{r=0}^1 r^3 e^{-r^2} \, dr = \pi \int_{r=0}^1 r^3 e^{-r^2} \, dr = \pi \left(\frac{1}{2} - e^{-1} \right) \approx 0.415$$

$$b) \quad I_1 = \int_{\varphi=0}^{\pi/2} \int_{r=0}^{\cos^2 \varphi} r \, dr \, d\varphi = \frac{1}{2} \int_{\varphi=0}^{\pi/2} \cos^4 \varphi \, d\varphi = \frac{1}{16} \int_{\varphi=0}^{\pi/2} (\cos(4\varphi) + 4 \cos(2\varphi) + 3) \, d\varphi = \frac{3}{32} \pi \approx 0.295$$

7. Doppelintegrale in der Volumenberechnung

a) $f(x, y) = 2 + \sin x \cdot \sin y$, $A_f: -\pi \leq x, y \leq \pi$

$$V = \iint_A (2 + \sin x \cdot \sin y) dx dy = \int_{x=-\pi}^{\pi} \int_{y=-\pi}^{\pi} (2 + \sin x \cdot \sin y) dx dy = 4\pi \int_{y=-\pi}^{\pi} dy = 8\pi^2 \approx 78.96 \text{ VE}$$

$g(x, y) = 2 + \sin x \cdot \sin y$, $A_g: x^2 + y^2 \leq \pi^2$

$$\begin{aligned} V &= \iint_A (2 + \sin x \cdot \sin y) dx dy = \int_{x=-\pi}^{\pi} \int_{y=-\sqrt{\pi^2-x^2}}^{\sqrt{\pi^2-x^2}} (2 + \sin x \cdot \sin y) dy dx = \\ &= 4 \int_{x=-\pi}^{\pi} \sqrt{\pi^2 - x^2} dx = 2\pi^3 \approx 62.01 \text{ VE} \end{aligned}$$

b) $x^2 + y^2 = 9$, $z = 0$, $z = 9 - y$

$$V = \iint_A (9 - y) dx dy = 81\pi \approx 254.47 \text{ VE}$$

c) $y = x^2$, $y = 4$, $z = 3 + x + 2y$

$$V = \iint_A (3 + x + 2y) dx dy = \int_{x=-2}^2 \int_{y=x^2}^4 (3 + x + 2y) dy dx = \frac{416}{5} \approx 83.2 \text{ VE}$$

d) $z = 2 - 2x - y$, $x = 0$, $y = 0$, $z = 0$, $V = \frac{2}{3}$