

Übungsblatt 10

Gewöhnliche Differentialgleichungen

Gewöhnliche Differentialgleichungen, Grundbegriffe:

A1.

Prüfen Sie zu der gegebenen DGL, ob die gegebene Funktion die Lösung der DGL ist:

$$x = \ln y' + \sin y', \quad x = \ln t + \sin t, \quad y = t(1 + \sin t) + \cos t$$

Gewöhnliche Differentialgleichungen 1. Ordnung

Integration durch Trennung der Variablen

A2

1. $2(x-1)dx + 3y^2 dy = 0, \quad y(2) = 1$

2. $y' = x(y-1), \quad y(0) = \frac{1}{2}$

3. $y' + x^2(y-2) = 0, \quad y(0) = -2$

A3

1. $y' - x^2 = 0, \quad a) y(-3) = 1, \quad b) y(2) = 5$

2. $y' = x^2 + 4x - 3, \quad a) y(1) = -2, \quad b) y(-3) = 3$

3. $y' = 4 + y, \quad a) y(0) = 1, \quad b) y(1) = 3$

Integration durch Substitution

A4:

1. $y' = 4x - y$, a) $y(2) = 4$, b) $y(0) = -2$

2. $y' = x + y$, a) $y(0) = 2$, b) $y(-2) = 3$

Variation der Konstanten

A5

1. $y' + y = m e^{-nx}$, $y(0) = 1$,

a) $m = n = 1$, b) $m = 1$, $n = 2$, c) $m = 2$, $n = 1$

2. $y' + \frac{y}{x} = \sin(mx)$, $y(\pi) = 1$, a) $m = 1$, b) $m = 2$

3. $xy' + my = x$, $y(1) = -1$, a) $m = 1$, b) $m = 2$, c) $m = 4$

4. $xy' + my = x^2$, $y(1) = 0$, a) $m = 2$, b) $m = -2$

5. $xy' - y = x^m$, $y(1) = 1$, a) $m = 1$, b) $m = 2$, c) $m = 3$

Lösungen – Übungsblatt 10:

1. Gewöhnliche Differentialgleichungen, Grundbegriffe:

$$x = \ln y' + \sin y', \quad x = \ln t + \sin t, \quad y = t(1 + \sin t) + \cos t$$

$$y' = y'_x = \frac{y'(t)}{x'(t)}$$

$$x'(t) = \frac{1}{t} + \cos t, \quad y'(t) = 1 + \sin t - \sin t + t(1 + \cos t) = 1 + t \cos t$$

$$y' = \frac{1 + t \cos t}{\frac{1}{t} + \cos t} = \frac{1 + t \cos t}{\frac{1}{t}(1 + t \cos t)} = t, \quad t > 0$$

$$\Rightarrow \ln t + \sin t = \ln t + \sin t$$

Lsg.

2.

Gewöhnliche Differentialgleichungen 1. Ordnung

Integration durch Trennung der Variablen

$$1. \quad 2(x-1)dx + 3y^2 dy = 0, \quad y(2) = 1$$

$$2 \int (x-1) dx + 3 \int y^2 dy = 0, \quad (x-1)^2 + y^3 = C$$

$$\text{Allgemeine Lösung: } (x-1)^2 + y^3 = C$$

$$y(2) = 1 : \quad C = 2, \quad (x-1)^2 + y^3 = 2$$

$$2. \quad y' = x(y-1), \quad y(0) = \frac{1}{2}$$

$$\int \frac{dy}{y-1} = \int x dx, \quad \ln |y-1| = \frac{x^2}{2} + \ln |C|, \quad \ln \left| \frac{y-1}{C} \right| = \frac{x^2}{2}$$

$$\text{Allgemeine Lösung: } y = e^{\frac{x^2}{2}} + C$$

$$y(0) = \frac{1}{2} : \quad C = -\frac{1}{2}, \quad y = e^{\frac{x^2}{2}} - \frac{1}{2}$$

Noch zu Lsg.2:

$$3. \quad y' + x^2(y - 2) = 0, \quad y(0) = -2$$

$$\int \frac{dy}{y-2} = - \int x^2 dx, \quad \ln |y-2| = -\frac{x^3}{3} + \ln |C|$$

$$\text{Allgemeine Lösung: } y = 2 + C e^{-\frac{x^3}{3}}$$

$$y(0) = -2 : C = -4, \quad y = 2 - 4 e^{-\frac{x^3}{3}}$$

Lsg.

3.

$$1. \quad y' - x^2 = 0, \quad a) \quad y(-3) = 1, \quad b) \quad y(2) = 5$$

$$\text{Allgemeine Lösung: } y = \frac{x^3}{3} + C$$

$$a) \quad y(-3) = 1 : y = \frac{x^3}{3} + 10, \quad b) \quad y(2) = 5 : y = \frac{x^3}{3} + \frac{7}{3}$$

$$2. \quad y' = x^2 + 4x - 3, \quad a) \quad y(1) = -2, \quad b) \quad y(-3) = 3$$

$$\text{Allgemeine Lösung: } y = \frac{x^3}{3} + 2x^2 - 3x + C$$

$$a) \quad y(1) = -2 : y = \frac{x^3}{3} + 2x^2 - 3x - \frac{4}{3}$$

$$b) \quad y(-3) = 3 : y = \frac{x^3}{3} + 2x^2 - 3x - 15$$

$$3. \quad y' = 4 + y, \quad a) \quad y(0) = 1, \quad b) \quad y(1) = 3$$

$$\text{Allgemeine Lösung: } y = C e^x - 4$$

$$a) \quad y(0) = 1 : y = 5 e^x - 4, \quad b) \quad y(1) = 3 : y = \frac{7}{e} e^x - 4 = 7 e^{x-1} - 4$$

Lsg.

Integration durch Substitution

4.

1. $y' = 4x - y$, a) $y(2) = 4$, b) $y(0) = -2$

$$u = 4x - y, \quad \frac{du}{dx} = 4 - \frac{dy}{dx} \Leftrightarrow u' = 4 - y'$$

$y' = 4x - y = u$ in Eq. $u' = 4 - y'$ einsetzen: $u' = 4 - u$

$$\frac{du}{dx} = 4 - u, \quad \int \frac{du}{4 - u} = \int dx, \quad \ln(4 - u) = x + \ln C, \quad 4 - u = C e^x$$

weiter folgt die Rücksubstitution

Allgemeine Lösung: $y = C e^{-x} + 4x - 4$

a) $y(2) = 4$: $y = 4x - 4$, b) $y(0) = -2$: $y = 2e^{-x} + 4x - 4$

2. $y' = x + y$, a) $y(0) = 2$, b) $y(-2) = 3$

Allgemeine Lösung: $y = C e^x - x - 1$

a) $y(0) = 2$: $y = 3e^x - x - 1$, b) $y(-2) = 3$: $y = 2e^{x+2} - x - 1$

Lsg.

Variation der Konstanten

5.

1. a) $m = n = 1$: $y' + y = e^{-x}$, $y = (x + C) e^{-x}$

$y(0) = 1$: $y = (1 + x) e^{-x}$

b) $m = 1, n = 2$: $y' + y = e^{-2x}$, $y = (-e^{-x} + C) e^{-x}$

$y(0) = 1$: $y = (2 - e^{-x}) e^{-x}$

c) $m = 2, n = 1$: $y' + y = 2e^{-x}$, $y = (2x + C) e^{-x}$

$y(0) = 1$: $y = (1 + 2x) e^{-x}$

2. a) $m = 1 : y' + \frac{y}{x} = \sin x, \quad y = \frac{1}{x} (\sin x - x \cos x + C)$
 $y(\pi) = 1 : y = \frac{1}{x} (\sin x - x \cos x)$
b) $m = 2 : y' + \frac{y}{x} = \sin(2x), \quad y = \frac{1}{4x} (\sin(2x) - 2x \cos(2x) + 4C)$
 $y(\pi) = 1 : y = \frac{1}{4x} \sin(2x) - \frac{1}{2} \cos(2x) + \frac{3\pi}{2x}$
3. a) $m = 1 : xy' + y = x, \quad y = \frac{x}{2} - \frac{C}{x}, \quad y(1) = -1 : y = \frac{x}{2} - \frac{3}{2x}$
b) $m = 2 : xy' + 2y = x, \quad y = \frac{x}{3} + \frac{C}{x^2}, \quad y(1) = -1 : y = \frac{x}{3} - \frac{4}{3x^2}$
c) $m = 4 : xy' + 4y = x, \quad y = \frac{x}{5} + \frac{C}{x^4}, \quad y(1) = -1 : y = \frac{x}{5} - \frac{6}{5x^4}$
4. a) $m = 2 : xy' + 2y = x^2, \quad y = \frac{x^2}{4} + \frac{C}{x^2}, \quad y(1) = 0 : y = \frac{x^2}{4} - \frac{1}{4x^2}$
b) $m = -2 : xy' - 2y = x^2, \quad y = (\ln x + C)x^2, \quad y(1) = 0 : y = x^2 \ln x$
5. a) $m = 1 : xy' - y = x, \quad y = (\ln x + C)x, \quad y(1) = 1 : y = (\ln x + 1)x$
b) $m = 2 : xy' - y = x^2, \quad y = (x + C)x, \quad y(1) = 1 : y = x^2$
c) $m = 3 : xy' - y = x^3, \quad y = \frac{x^3}{2} + Cx, \quad y(1) = 1 : y = \frac{x^3}{2} + \frac{x}{2}$